Math 612 part 4 — Spectral Sequences

Note Title 10/29/2014

Important tool for calculating cohomology in the presence of an additional structure on the chain cx: filtration.

Uses:
- Simplifying pf of deRham ⟷ Čech
- reproving Künneth
- generalizing to fiber bundles: Leray-Hirsch Thm, Leray-Serre spectral sequence.

Setup: (K, D) complex: K = abelian group, $D: K \to K$ with $D^2 = 0$.
$\rightsquigarrow H(K,D) = \ker D / \operatorname{im} D$.

Note: most familiar complexes are graded:

$K = \oplus K_n$, $D: K_n \to K_{n-1}$ so $\to K_{n+1} \xrightarrow{D} K_n \xrightarrow{D} K_{n-1} \to \cdots$ say "D has degree -1"

$D: K \to K$ given by $D = \oplus D$

or $K = \oplus K^n$, $D: K^n \to K^{n+1}$ so $\cdots \to K^{n-1} \xrightarrow{D} K^n \xrightarrow{D} K^{n+1} \to \cdots$ say "D has degree +1"

and again $D: K \to K$ given by $D = \oplus D$.

this will usually be the case.

$H(K,D) = \oplus H^n(K,D)$.

Def A subcomplex K' of K is a subgroup with $D(K') \subset K'$.

Useless ex: $\bigoplus_{n \geq N} K^n$.

A descending filtration of K is a nested sequence of subcomplexes

$$K = \mathcal{F}^0(K) \supseteq \mathcal{F}^1(K) \supseteq \mathcal{F}^2(K) \supseteq \cdots$$

Convention: extend to $\mathcal{F}^n(K) = K$ for $n < 0$.

This makes (K, D) a filtered complex.

Note $D: \mathcal{F}^m(K) \circlearrowleft$, $D: \mathcal{F}^{m+1}(K) \circlearrowleft$ $\Rightarrow$ $D: \mathcal{F}^m(K)/\mathcal{F}^{m+1}(K) \circlearrowleft$

The associated graded complex to this filtered complex is

$$\left(\operatorname{Gr} K = \bigoplus_{m=0}^{\infty} \mathcal{F}^m(K)/\mathcal{F}^{m+1}(K) \ , \ D = \oplus D \right)$$

Note: usually K is graded, and so is the filtration:

$\mathcal{F}^m(K) = \oplus \mathcal{F}^m(K^n)$, $K^n = \mathcal{F}^0(K^n) \supseteq \mathcal{F}^1(K^n) \supseteq \cdots$, $D: \mathcal{F}^m(K^n) \to \mathcal{F}^m(K^{n+1})$

$$K = \mathcal{F}^0 K \supset \mathcal{F}^1 K \supset \mathcal{F}^2 K \supset \cdots$$

$$\begin{tikzcd}
 & {} & & {} & & {} & \\
K^{n+1} = & \mathcal{F}^0 K^{n+1} \arrow[u] & \supset & \mathcal{F}^1 K^{n+1} \arrow[u] & \supset & \mathcal{F}^2 K^{n+1} \arrow[u] & \supset \cdots \\
K^n = & \mathcal{F}^0 K^n \arrow[u, "D"] & \supset & \mathcal{F}^1 K^n \arrow[u, "D"] & \supset & \mathcal{F}^2 K^n \arrow[u, "D"] & \supset \cdots \\
 & {} \arrow[u] & & {} \arrow[u] & & {} \arrow[u] &
\end{tikzcd}$$

$\mathcal{F}^m(K^n) = \mathcal{F}^m(K) \cap K^n$.

$\oplus$ columns:

$\left(K = \oplus K^n\right) \supset \left(\mathcal{F}^1 K = \oplus \mathcal{F}^1(K^n)\right) \supset \left(\mathcal{F}^2 K = \oplus \mathcal{F}^2(K^n)\right) \supset \cdots$

Each filtered piece has a grading, and each graded piece has a filtration.

Note then that the associated graded complex is also graded:

$$(\operatorname{Gr} K)^n = \bigoplus_{m=0}^{\infty} \mathcal{F}^m(K^n)/\mathcal{F}^{m+1}(K^n).$$

Complex

(K, D), K abelian grp

$D : K \to K$, $D^2 = 0$

$H(K, D) = \ker D / \operatorname{im} D$

Subcomplex: $K' \subset K$ with $D(K') \subset K'$.

(Descending) filtration: subcomplexes

$$\begin{tikzcd}[column sep=tiny] K \arrow[d, phantom, "\shortparallel"] & = & K \arrow[d, phantom, "\shortparallel"] & = & K \arrow[d, phantom, "\shortparallel"] & \supseteq & \mathcal{F}^1 K & \supseteq & \mathcal{F}^2 K & \supseteq \cdots \\ \mathcal{F}^{-2}K & & \mathcal{F}^{-1}K & & \mathcal{F}^0 K \end{tikzcd}$$

This is a filtered complex.

Associated graded complex:

$D : \mathcal{F}^m(K)/\mathcal{F}^{m+1}(K) \circlearrowleft$

$\left(Gr\, K = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}^m K / \mathcal{F}^{m+1} K, D = \oplus D\right)$

Graded Complex

$K = \oplus K^n$, $D : K^n \to K^{n+1}$

$\cdots \to K^{n-1} \xrightarrow{D} K^n \xrightarrow{D} K^{n+1} \to \cdots$

$D : K \to K$ defined by $D = \oplus D$

$H(K, D) = \bigoplus_n H^n(K, D)$

Filtration on each K^n and $D : \mathcal{F}^m K^n \to \mathcal{F}^m K^{n+1}$

$$\begin{tikzcd}[column sep=small] & \uparrow & & \uparrow & & \uparrow \\ K^{n+1} = & \mathcal{F}^0 K^{n+1} & \supseteq & \mathcal{F}^1 K^{n+1} & \supseteq & \mathcal{F}^2 K^{n+1} & \supseteq \cdots \\ K^n = & \mathcal{F}^0 K^n \arrow[u, "D"] & \supseteq & \mathcal{F}^1 K^n \arrow[u, "D"] & \supseteq & \mathcal{F}^2 K^n \arrow[u, "D"] & \supseteq \cdots \\ & \uparrow & & \uparrow & & \uparrow \end{tikzcd}$$

$\oplus$:

$\left(K = \oplus K^n\right) \supseteq \left(\mathcal{F}^1 K = \oplus \mathcal{F}^1 K^n\right) \supseteq \left(\mathcal{F}^2 K = \oplus \mathcal{F}^2 K^n\right) \supseteq \cdots$

This is a graded filtered complex.

Each filtered piece has a grading,

and each graded piece has a filtration.

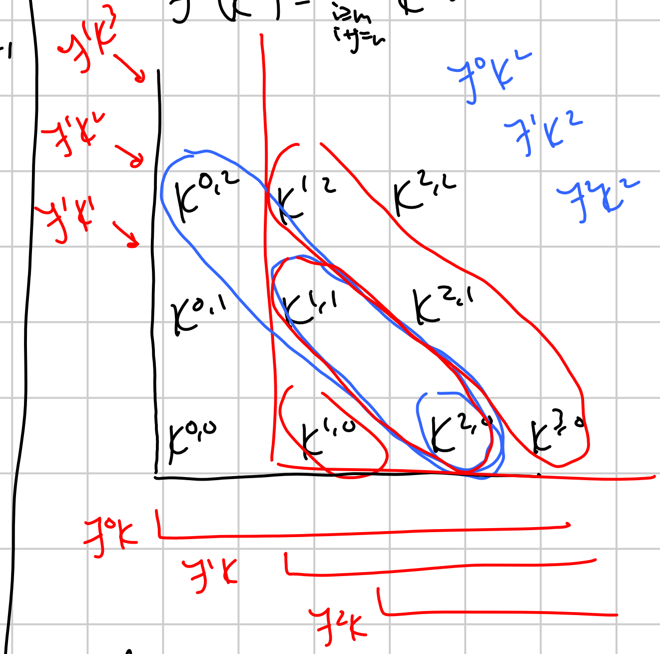
Example

$\left(K = \oplus K^{i,j}, \delta, d\right)$, $D = \delta + (-1)^i d$

$K^n = \bigoplus_{i+j=n} K^{i,j}$

$\mathcal{F}^m(K) = \bigoplus_{i \geq m} K^{i,j}$

$\mathcal{F}^m(K^n) = \bigoplus_{\substack{i \geq m \\ i+j=n}} K^{i,j}$



Assoc. graded:

$\mathcal{F}^m K / \mathcal{F}^{m+1} K \cong \bigoplus_j K^{m,j}$

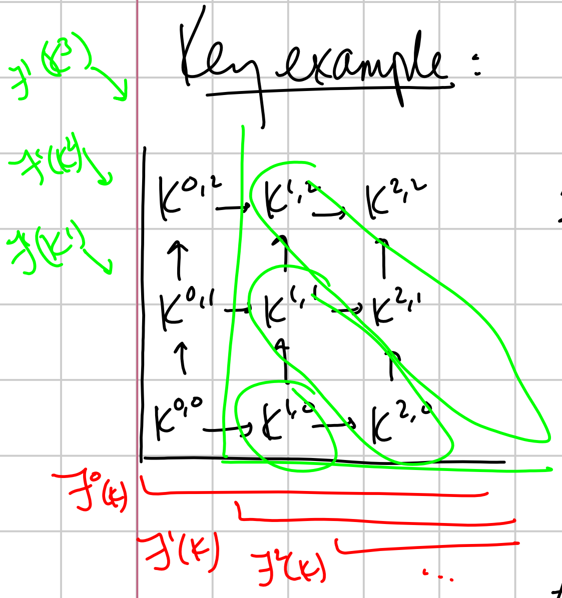
$\Rightarrow Gr\, K = \bigoplus_{m=0}^{\infty} \left(\bigoplus_j K^{m,j}\right) \cong K$

but induced diff on $Gr\, K$ is

$D = (-1)^m d$.

Key example: $(K=\bigoplus K^{i,j},\delta,d)$ double cx, $D=\delta+(-1)^i d$,

$$K^n=\bigoplus_{i+j=n} K^{i,j}.$$



Filtration:

$\mathcal{F}^m(K)=\bigoplus_{\substack{i,j\\ i\geq m}} K^{i,j}$ is a subcomplex of (K,D)

graded: $\mathcal{F}^m(K^n)=\bigoplus_{\substack{i\geq m\\ i+j=n}} K^{i,j}$

$$\mathcal{F}^m(K)=\bigoplus_n \mathcal{F}^m(K^n)$$

Associated graded: $\mathcal{F}^m(K)/\mathcal{F}^{m+1}(K)\cong\bigoplus_j K^{m,j}$

$$\Rightarrow \mathrm{Gr}\,K=\bigoplus_{m=0}^{\infty}\Big(\bigoplus_j K^{m,j}\Big)$$

Looks exactly like K: but the induced differential on $\mathrm{Gr}\,K$ is $(-1)^m d$.

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Idea: Sometimes $H^*(\mathrm{Gr}\,K,D)$ is easy to compute, even if $H^*(K,D)$ isn't.

Here: for $K^{*,*}=\check{C}^*(\mathcal{U},\Omega^*)$ with $\mathcal{U}$ = good cover,

$$H^*(\mathrm{Gr}\,K,D)\cong\bigoplus_m \check{C}^m(\mathcal{U},\underline{\mathbb{R}}).$$

What's $H^*(K,D)$ in this case? It's the homology of this homology:

$$H^*(K,D)\cong H^*(\check{C}^*(\mathcal{U},\underline{\mathbb{R}}),\delta)\cong H^*\big(H^*(\mathrm{Gr}\,K,D),\delta\big).$$

Def A spectral sequence is a sequence of complexes $(E_1,d_1),(E_2,d_2),\dots$ with $E_k=H(E_{k-1},d_{k-1})$.

If at some point $d_k=d_{k+1}=\dots=0$ then $E_{k+1}=H(E_k,0)=E_k$ etc. and we write $E_\infty:=E_k=E_{k+1}=\dots$ and say that the spectral sequence converges to E_∞.

In our ex: $E_1 = H^*(\mathrm{Gr}\, K, D) = C^*(\mathcal{U}, \underline{\mathbb{R}})$ $\quad d_1 = \delta$

$K = C^*(\mathcal{U}, \Omega^*)$ $\quad E_2 = \check{H}^*(\mathcal{U}, \underline{\mathbb{R}})$ $\quad d_2 = 0$

$\vdots$

$E_\infty = \check{H}^*(\mathcal{U}, \underline{\mathbb{R}}) = H^*(K, D)$.

We'll see more generally: $\exists$ s.s. with $E_1 = H^*(\mathrm{Gr}\, K, D)$ converging to $E_\infty = H^*(K, D)$, for a large class of filtered cxs.

Main technical tool: exact couples.

<u>Def</u> (Massey) An <u>exact couple</u> is an exact triangle of abelian groups

ker i = im k here etc.

$$\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

(often A, E are graded and k raises degree by 1).

From this, we can construct a <u>derived exact couple</u>

$$\begin{tikzcd}
A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\
& E' \arrow[ul, "k'"] &
\end{tikzcd}$$

as follows:

write $d = jk : E \to E$

$$\begin{tikzcd}
E \arrow[r, "k"] \arrow[rr, bend right, "d"'] & A \arrow[r, "j"] & E
\end{tikzcd}$$

and note $d^2 = jkjk = 0$.

Define

$$\begin{tikzcd}
A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\
& E' \arrow[ul, "k'"] &
\end{tikzcd}$$

by $A' = i(A)$, $E' = H(E, d)$.

Maps: $\underline{A' \xrightarrow{i'} A'}$: $i(A) \xrightarrow{i} i(A)$ gives i': $i'(ia) = iia$.

$\underline{A' \xrightarrow{j'} E'}$: define $j'(ia) = [ja] \in H(E, d)$.

Well-defined: $dja = 0$ and if $ia = i\tilde{a}$ then $a - \tilde{a} = kb \Rightarrow ja - j\tilde{a} = jkb = db$.

$\underline{E' \xrightarrow{k'} A'}$: $k'[b] = kb$.

Well-defined: $db = 0 \Rightarrow jkb = 0 \Rightarrow kb \in \operatorname{im} i$;

if $[b] = 0$ then $b = d\tilde{b}$ so $kb = kd\tilde{b} = kjk\tilde{b} = 0$.

Lemma:
$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$
is an exact triangle.

Pf Note $j'i' = k'j' = i'k' = 0$.

Let's check exactness at E'. If $k'[b]=0$ then $kb=0$

$\Rightarrow \exists a$ with $b = ja \Rightarrow j'(ia) = [ja] = [b]$.

Exactness at other corners is similar. □

So: given
$$\begin{tikzcd} A_1 \arrow[rr] & & A_1 \arrow[dl] \\ & E_1 \arrow[ul] & \end{tikzcd}$$
get
$$\begin{tikzcd} A_2 \arrow[rr] & & A_2 \arrow[dl] \\ & E_2 \arrow[ul] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} A_3 \arrow[rr] & & A_3 \arrow[dl] \\ & E_3 \arrow[ul] & \end{tikzcd} \rightsquigarrow \cdots$$
$E_2 = H(E_1, d_1)$ $E_3 = H(E_2, d_2)$

Where to start?

(K,D) filtered cx.

- $A_0 := \bigoplus_{m\in\mathbb{Z}} \mathcal{F}^m(K) = \cdots$

$$\begin{tikzcd} & \mathcal{F}^{-1}(K) \arrow[d, phantom, "\parallel"] & \mathcal{F}^0(K) \arrow[d, phantom, "\parallel"] & & & \\ \cdots \oplus & K \arrow[d, "i"] \oplus & K \arrow[d, "i"] \oplus & \mathcal{F}^1(K) \arrow[d, "i"] \oplus & \mathcal{F}^2(K) \arrow[d, "i"] \oplus & \mathcal{F}^3(K) \arrow[d, "i"] \oplus \cdots \\ \cdots \oplus & K \oplus & K \oplus & K \oplus & \mathcal{F}^1(K) \oplus & \mathcal{F}^2(K) \oplus \cdots \end{tikzcd}$$

$(\mathcal{F}^m(K), D)$ is a cx $\Rightarrow (A_0, D = \oplus D)$ is a cx.

Also, $\mathcal{F}^m(K) \to \mathcal{F}^{m-1}(K)$ induces $i: A_0 \to A_0$:

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- $E_0 := \mathrm{Gr}\, K = \oplus \mathcal{F}^m(K)/\mathcal{F}^{m+1}(K)$

There's a short exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & A_0 \arrow[r, "i"] \arrow[d, "D"] & A_0 \arrow[r] \arrow[d, "D"] & E_0 \arrow[r] \arrow[d, "D"] & 0 \\ 0 \arrow[r] & A_0 \arrow[r] & A_0 \arrow[r] & E_0 \arrow[r] & 0 \end{tikzcd}$$
$(\mathcal{F}^m \to \mathcal{F}^{m-1} \to \mathcal{F}^{m-1}/\mathcal{F}^m)$

Leading to an exact triangle

$$\begin{tikzcd} H(A_0,D) \arrow[r, "i_*"] & H(A_0,D) \arrow[r] & H(E_0,D) \arrow[ll, bend left] \end{tikzcd}$$

Recap:

(K, D) filtered cx : $\cdots = \overset{\overset{F^{-1}(K)}{\shortparallel}}{K} = \overset{\overset{F^{0}(K)}{\shortparallel}}{K} \supseteq F^1(K) \supseteq F^2(K) \supseteq \cdots$

Often will be a graded filtered cx: each graded piece K^n is filtered and $D: F^m(K^n) \to F^m(K^{n+1})$.

We'll prove:

$F^\ell(K) = 0$ for suff large ℓ.

Prop (K, D) filtered cx, filtration of finite length. Then $\exists$ spectral sequence $(E_1, d_1), (E_2, d_2), \ldots$ with $E_1 = H(\operatorname{Gr} K)$, converging to $\operatorname{Gr} H(K)$, the associated graded gp to $H(K)$.

Strategy: exact couple

$$\begin{tikzcd}[column sep=small] A_1 \arrow[r] & A_1 \arrow[dl] & & A_2 \arrow[r] & A_2 \arrow[dl] & & & A_\infty \arrow[r] & A_\infty \arrow[dl] \\ E_1 \arrow[u] \arrow[rrr, squiggly] & & & E_2 \arrow[u] \arrow[rr, squiggly] & & \cdots & & E_\infty = \operatorname{Gr} H(K) \arrow[u] & \end{tikzcd}$$

- $A_0 := \bigoplus_{m \in \mathbb{Z}} F^m(K)$, $E_0 := \operatorname{Gr} K = \bigoplus F^m(K)/F^{m+1}(K)$

$$\begin{tikzcd} \vdots & \vdots & \vdots \\ \oplus & \oplus & \oplus \\ K \arrow[r, "="] & K \arrow[r] & 0 \\ \oplus & \oplus & \oplus \\ K \arrow[r, "="] & K \arrow[r] & 0 \\ \oplus & \oplus & \oplus \\ F^1K \arrow[r, hook, "i"] & K \arrow[r] & K/F^1K \\ \oplus & \oplus & \oplus \\ F^2K \arrow[r, hook, "i"] & F^1K \arrow[r] & F^1K/F^2K \\ \oplus & \oplus & \oplus \\ F^3K \arrow[r, hook, "i"] & F^2K \arrow[r] & F^2K/F^3K \\ \oplus & \oplus & \oplus \\ \vdots & \vdots & \vdots \\ \circled{A_0} & \circled{A_0} & \circled{E_0} \end{tikzcd}$$

Get short exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & A_0 \arrow[r, "i"] \arrow[d, "D"'] & A_0 \arrow[r] \arrow[d, "D"'] & E_0 \arrow[r] \arrow[d, "D"'] & 0 \\ 0 \arrow[r] & A_0 \arrow[r, "i"] & A_0 \arrow[r] & E_0 \arrow[r] & 0 \end{tikzcd}$$

leading to an exact triangle

$$\begin{tikzcd} H(A_0, D) \arrow[r] & H(A_0, D) \arrow[r] & H(E_0, D) \arrow[ll, bend left] \end{tikzcd}$$

this is the connecting homomorphism.

In the graded case, this is: $A_0 = \bigoplus_n \mathcal{F}^m(K^n)$, $E_0 = \bigoplus_n \mathcal{F}^m(K^n)/\mathcal{F}^{m+1}(K^n)$ are graded cxs.

$\rightsquigarrow$ LES in cohomology

$$\cdots \to H^n(A_0,D) \to H^n(A_0,D) \to H^n(E_0,D) \to H^{n+1}(A_0,D) \to \cdots$$

which can be wrapped into

$$\begin{tikzcd}
H(A_0,D) = \bigoplus_n H^n(A_0,D) \arrow[rr] & & \bigoplus_n H^n(A_0,D) = H(A_0,D) \arrow[dl] \\
& \bigoplus_n H^n(E_0,D) = H(E_0,D) \arrow[ul] &
\end{tikzcd}$$

So we can start with the exact couple
$$\begin{tikzcd}
A_1 \arrow[rr] & & A_1 \arrow[dl] \\
& E_1 \arrow[ul] &
\end{tikzcd},$$

$A_1 = H(A_0,D) = \bigoplus_m H(\mathcal{F}^m(K),D)$

$E_1 = H(E_0,D) = H(\mathrm{Gr}\,K,D) = \bigoplus_m H(\mathcal{F}^m(K)/\mathcal{F}^{m+1}(K),D)$.

Combine A_1 and $i_1 = i_*: A_1 \to A_1$:

$$\begin{tikzcd}
A_1\text{: (direct sum of)} & \cdots & H(K) \arrow[l] & H(K) \arrow[l] \arrow[d, "j_1"'] & H(\mathcal{F}^1K) \arrow[l, "i_1"'] \arrow[d, "j_1"'] & H(\mathcal{F}^2K) \arrow[l, "i_1"'] \arrow[d, "j_1"'] & \cdots \arrow[l, "i_1"'] \\
\text{now add } E_1: & & & H(\mathcal{F}^0K/\mathcal{F}^1K) \oplus \arrow[ur, "k_1"] & H(\mathcal{F}^1K/\mathcal{F}^2K) \oplus \arrow[ur, "k_1"] & H(\mathcal{F}^2K/\mathcal{F}^3K) \oplus \arrow[ur] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "j_1"] & \rightsquigarrow & A_2 \arrow[rr, "i_2"] & & A_2 \arrow[dl] & \rightsquigarrow & A_3 \arrow[rr, "i_3"] & & A_3 \arrow[dl] & \rightsquigarrow \cdots \\
& E_1 \arrow[ul, "k_1"] & & & & E_2 \arrow[ul] & & & & E_3 \arrow[ul] & &
\end{tikzcd}$$

Let's concentrate on the A's.

$$\begin{tikzcd}
A_1 = \oplus\text{ terms in } \cdots & H(K) \arrow[l, "="'] \arrow[d, phantom, "\parallel"] & H(K) \arrow[l, "="'] \arrow[d, phantom, "\parallel"] & H(K) \arrow[l, "="'] \arrow[d, phantom, "\cup"] & H(\mathcal{F}^1K) \arrow[l, "i_*"'] \arrow[d, phantom, "\cup"] & H(\mathcal{F}^2K) \arrow[l, "i_*"'] \arrow[d, phantom, "\cup"] & \cdots \arrow[l, "i_*"'] & (\text{maps} = i_1) \\
\rightsquigarrow A_2 = i_1A_1 = \oplus\text{ terms in } \arrow[r, leftarrow, "="] & H(K) \arrow[d, phantom, "\parallel"] & H(K) \arrow[l, "="'] \arrow[d, phantom, "\cup"] & i_*H(\mathcal{F}^1K) \arrow[l, hook', "i_*"'] \arrow[d, phantom, "\cup"] & i_*H(\mathcal{F}^2K) \arrow[l, "i_*"'] \arrow[d, phantom, "\cup"] & i_*H(\mathcal{F}^3K) \arrow[l, "i_*"'] \arrow[d, phantom, "\cup"] & \cdots \arrow[l, "i_*"'] & (\text{maps} = i_2) \\
\rightsquigarrow A_3 = i_2A_2 = \oplus\text{ terms in } \arrow[r, leftarrow, "="] & H(K) & i_*H(\mathcal{F}^1K) \arrow[l, hook', "i_*"'] & i_*^2H(\mathcal{F}^2K) \arrow[l, hook', "i_*"'] & i_*^2H(\mathcal{F}^3K) \arrow[l, "i_*"'] & i_*^2H(\mathcal{F}^4K) \arrow[l, "i_*"'] & \cdots \arrow[l] & (\text{maps} = i_3)
\end{tikzcd}$$

↑ inclusion!

↑ inclusion since $i_*^2H(\mathcal{F}^2K) \subset i_*H(\mathcal{F}^1K) \subset H(K)$

$\vdots$

Now suppose filtration is <u>finite length</u>: $\mathcal{F}^\ell(K) \neq 0$, $\mathcal{F}^{\ell+1}(K) = 0$. Then

$$\begin{tikzcd} A_{\ell+1} = & {} & H(K) \arrow[l, "="'] & i_* H(\mathcal{F}^1 K) \arrow[l, hook', "i_*"'] & i_*^2 H(\mathcal{F}^2 K) \arrow[l, hook', "i_*"'] & \cdots \arrow[l, hook', "i_*"'] & i_*^\ell H(\mathcal{F}^\ell K) \arrow[l, hook', "i_*"'] & 0 \arrow[l] \end{tikzcd}$$

So

$$\begin{tikzcd} A_{\ell+1} \arrow[rr, hook, "i_*"] & & A_{\ell+1} \arrow[dl] \\ & E_{\ell+1} \arrow[ul, "0"] & \end{tikzcd}$$

and $A_{\ell+1} = A_{\ell+2} = \cdots =: A_\infty$

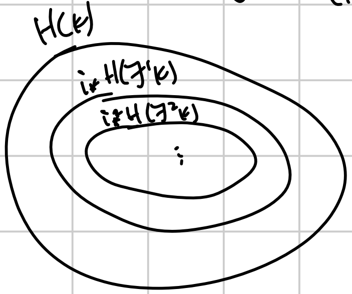
$d_{\ell+1} = 0 \Rightarrow E_{\ell+1} = E_{\ell+2} = \cdots =: E_\infty$.

What's E_∞? It's the cokernel of $i : A_\infty \hookrightarrow A_\infty$.

$$A_\infty = \cdots \leftarrow H(K) \hookleftarrow i_* H(\mathcal{F}^1 K) \hookleftarrow i_*^2 H(\mathcal{F}^2 K) \hookleftarrow \cdots$$

Better: give the group $H(K)$ a filtration: $H(K) = \mathcal{F}^0 H(K) \supset \mathcal{F}^1 H(K) \supset \cdots \supset \mathcal{F}^\ell H(K) \supset 0$

$\mathcal{F}^m H(K) := i_*^m H(\mathcal{F}^m K)$.



(think: $H(K) \xleftarrow{i_*} H(\mathcal{F}^1 K) \xleftarrow{i_*} H(\mathcal{F}^2 K) \xleftarrow{i_*} \cdots$)

Then $E_\infty \cong \bigoplus \left(\mathcal{F}^m H(K) / \mathcal{F}^{m+1} H(K) \right)$

= <u>associated graded group</u> to the filtered gp $H(K)$.

So we've proved:

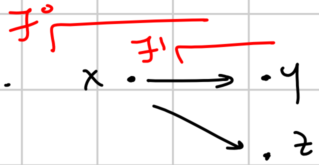
<u>Prop</u> (K, D) filtered cx, filtration of finite length. Then $\exists$ spectral sequence $(E_1, d_1), (E_2, d_2), \ldots$ with $E_1 = H(\mathrm{Gr}\,K)$, converging to <u>$\mathrm{Gr}\,H(K)$</u>, the associated graded gp to $H(K)$.

Note If K = vector space then $H(K)$ = v.s. $\Rightarrow$

$$H(K) \cong \left(H(K)/\mathcal{F}^1 H(K)\right) \oplus \left(\mathcal{F}^1/\mathcal{F}^2\right) \oplus \cdots \oplus \left(\mathcal{F}^\ell/\mathcal{F}^{\ell+1}\right)$$

so $E_\infty \cong H(K)$. **Not** true in general : see f/w.

Ex 1.



$K = \mathcal{F}^0 = \mathbb{R}\langle x, y, z\rangle$ $\qquad dx = y+z$

$\mathcal{F}^1 = \langle y, z\rangle$ $\qquad dy = dz = 0.$

$H(K) = \langle y\rangle$, $H(\mathcal{F}^1) = \langle y, z\rangle$

$$\begin{tikzcd}
 & & & H(\mathcal{F}^0) & H(\mathcal{F}^1) & \\
A_1 = & \cdots & \langle y\rangle \arrow[l, Rightarrow, "="'] & \langle y\rangle \arrow[l, "="'] \arrow[d, "j=0"'] & \langle y,z\rangle \arrow[l, "i"'] \arrow[d, "j=i\partial"] & 0 \arrow[l] \\
 & & & \langle x\rangle \arrow[ur, "k"] \arrow[r, phantom, "\oplus"] & \langle y,z\rangle & \\
 & & & H(\mathcal{F}^0/\mathcal{F}^1) & H(\mathcal{F}^1/\mathcal{F}^2) &
\end{tikzcd}$$

$i(y) = y$, $i(z) = -y$

$E_1 = \langle x, y, z\rangle$

$k(x) = y+z$

$j : H(\mathcal{F}^m) \longrightarrow H(\mathcal{F}^m/\mathcal{F}^{m+1})$, $k : H(\mathcal{F}^m/\mathcal{F}^{m+1}) \to H(\mathcal{F}^{m+1})$

$$\begin{tikzcd}
 & y+z \arrow[r, red] & y+z & & \\
0 \arrow[r] & \mathcal{F}^{m+1} \arrow[r] & \mathcal{F}^m \arrow[r] & \mathcal{F}^m/\mathcal{F}^{m+1} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{F}^{m+1} \arrow[r] & \mathcal{F}^m \arrow[r] \arrow[u, "\partial"] & \mathcal{F}^m/\mathcal{F}^{m+1} \arrow[r] & 0 \\
 & & x \arrow[r, red] \arrow[uuu, red] & x &
\end{tikzcd}$$

$d_1(x) = jk(x) = y+z$

$\Rightarrow E_2 = H(E_1, d_1) = \langle y\rangle \quad = H(K).$

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Review: (K, D) filtered cx, finite filtration

$$K = \mathcal{F}^0 K \supset \mathcal{F}^1 K \supset \cdots \supset \mathcal{F}^\ell K \supset \mathcal{F}^{\ell+1} K = 0.$$

$\exists$ spectral sequence $(E_1, d_1), (E_2, d_2), \ldots \longrightarrow E_\infty$ with $E_1 = H(Gr\, K, D)$ and $E_\infty = Gr\, H(K)$.

$$\begin{tikzcd} A_1 \arrow[rr, "i"] & & A_1 \arrow[dl] \\ & E_1 \arrow[ul] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} A_2 \arrow[rr, "i"] & & A_2 \arrow[dl] \\ & E_2 \arrow[ul] & \end{tikzcd} \rightsquigarrow \cdots$$

$$\begin{tikzcd} A_1 = & \cdots & H(K) \arrow[l, "="'] & H(K) \arrow[l, "="'] \arrow[d, "j"'] & H(\mathcal{F}^1 K) \arrow[l, "i"'] \arrow[d, "j"'] & H(\mathcal{F}^2 K) \arrow[l, "i"'] \arrow[d, "j"'] & \cdots \arrow[l, "i"'] \\ E_1 : & & & H(K/\mathcal{F}^1 K) \arrow[ur, "k"] & H(\mathcal{F}^1 K/\mathcal{F}^2 K) \arrow[ur, "k"] & H(\mathcal{F}^2 K/\mathcal{F}^3 K) \arrow[ur, "k"] & \end{tikzcd}$$

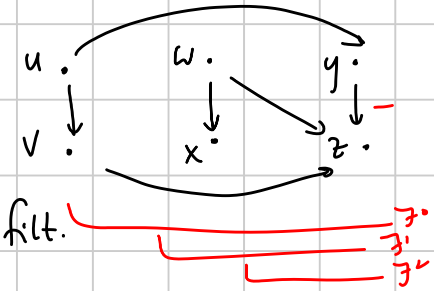
Filtration on $H(K)$: $H(K) = \mathcal{F}^0 H(K) \supset \mathcal{F}^1 H(K) \supset \mathcal{F}^2 H(K) \supset \cdots$

$\mathcal{F}^m H(K) = i^m H(\mathcal{F}^m K)$. $\quad Gr\, H(K) = \oplus \mathcal{F}^m H(K) / \mathcal{F}^{m+1} H(K)$.

Note: if $H(K)$ = vector space then $Gr\, H(K) = K/\mathcal{F}^1 \oplus \mathcal{F}^1/\mathcal{F}^2 \oplus \cdots \cong K$.

Not true in general! ex: $H(K) = \mathbb{Z}$, $\mathcal{F}^1 H(K) = 2\mathbb{Z}$: $Gr\, H(K) \cong (\mathbb{Z}/2) \oplus \mathbb{Z} \ncong \mathbb{Z}$.

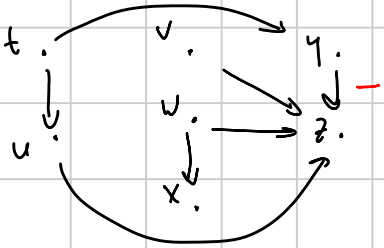
Ex 2a



$du = v + y$, $dv = z$, $dw = x + z$, $dy = -z$

Choose the filt. $\Rightarrow E_1 = 0 \Rightarrow H(K) = 0$

Ex 2b



$E_1 = \langle v \rangle$, $d_1^2 = 0 \Rightarrow d_1 = 0$

$\therefore H(K) \cong \mathbb{R}$.

Adding in the grading

Now say $K = \oplus K^n$ is a graded, filtered complex.

Prop K graded filtered cx s.t. for each n, the filtration on K^n has finite length. Then $\exists$ spectral sequence $(E_1, d_1), \dots$ with $E_1 = H^*(\mathrm{Gr}\, K)$ that converges to $\mathrm{Gr}\, H^*(K)$.
(in particular $E_1^n = H^n(\mathrm{Gr}\, K)$ converges to $\mathrm{Gr}\, H^n(K)$).

Pf Say
$$\begin{tikzcd} A \arrow[rr] & & A \arrow[dl] \\ & E \arrow[ul] & \end{tikzcd}$$
is graded if A, E are graded and
$$\begin{tikzcd} {} \arrow[rr, "\Delta dm=0"] & & {} \arrow[dl, "\Delta=0"] \\ & {} \arrow[ul, "\Delta=1"] & \end{tikzcd}$$

If
$$\begin{tikzcd} A_1 \arrow[rr] & & A_1 \arrow[dl] \\ & E_1 \arrow[ul] & \end{tikzcd}$$
is graded then so are the derived couples.

In our setting we have $0 \to A_0 \to A_0 \to E_0 \to 0$ grading-preserving

$$\Rightarrow \quad \cdots \to H^n(A_0) \to H^n(A_0) \to H^n(E_0) \xrightarrow{+1} H^{n+1}(A_0) \to \cdots$$

so
$$\begin{tikzcd} A_1 \arrow[rr] & & A_1 \arrow[dl] \\ & E_1 \arrow[ul, "\alpha"] & \end{tikzcd}$$
is graded.

Now

$$A_\ell = \cdots \hookleftarrow H(K) \hookleftarrow iH(\mathcal{F}^1 K) \hookleftarrow i^2 H(\mathcal{F}^2 K) \hookleftarrow \cdots \hookleftarrow i^{\ell-1} H(\mathcal{F}^{\ell-1} K) \leftarrow i^{\ell-1} H(\mathcal{F}^\ell K) \leftarrow \cdots ;$$

Suppose $\ell > \ell(n)$ where $\ell(n) =$ length of filtration on K^n
i.e. $\mathcal{F}^m(K^n) = 0$ for $m > \ell(n)$.

Then $\ell > \ell(n) \Rightarrow$

$$A_\ell^n = \cdots \hookleftarrow H(K^n) \hookleftarrow iH(\mathcal{F}^1 K^n) \hookleftarrow \cdots \hookleftarrow i^{\ell(n)} H(\mathcal{F}^{\ell(n)} K)$$

and $i : A_\ell^n \to A_\ell^n$ is an injection $\Rightarrow$ from
$$\begin{tikzcd} A_\ell^n \arrow[rr, "i"] & & A_\ell^n \\ & E_\ell^{n-1} \arrow[ul, "map=0"'] & \end{tikzcd}$$
, $d_\ell : E_\ell^{n-1} \to E_\ell^n$ is 0.

Thus if $\ell > \max(\ell(n), \ell(n+1))$, then

$$E_\ell^{n-1} \xrightarrow{d_\ell = 0} E_\ell^n \xrightarrow{d_\ell = 0} E_\ell^{n+1} \quad \text{so } E_\ell^n = E_{\ell+1}^n = \cdots = E_\infty^n.$$

Finally:

$$\begin{tikzcd} & & iH^n(\mathcal{F}^1K) \arrow[d, equal] & & i^{l(n)}H^n(\mathcal{F}^{l(n)}K) \arrow[d, equal] \\ A^n_\infty = & H^n(K) \arrow[l, hook'] & \mathcal{F}^1H^n(K) \arrow[l, hook', "i"'] & \cdots \arrow[l, hook', "i"'] & \mathcal{F}^{l(n)}H^n(K) \arrow[l, hook', "i"'] \end{tikzcd}$$

and

$$\begin{tikzcd} A^n_\infty \arrow[rr, hook] & & A^n_\infty \arrow[dl] \\ & E^n_\infty \arrow[ul] & \end{tikzcd} \Rightarrow \quad E^n_\infty = \operatorname{coker}\left(A^n_\infty \xrightarrow{i} A^n_\infty\right)$$

$$= \bigoplus \mathcal{F}^mH^n(K)/\mathcal{F}^{m+1}H^n(K). \qquad \square$$

The spectral sequence for a double complex

$\mathcal{F}^m(K) = \bigoplus_{i \geq m} K^{i,j}$, $\mathcal{F}^m(K^n) = \bigoplus_{\substack{i \geq m \\ i+j=n}} K^{i,j}$. Write $d' = (-1)^i d \Rightarrow D = \delta + d'$.

Note: $(K^{*,*}, \delta, d) \longrightarrow (K^*, D)$ satisfies the conditions of the Prop.

(K^n has a finite filtration for each n).

- $\underline{E_1} = H^*(\mathrm{Gr}\,K, \text{induced } D)$. $\mathrm{Gr}\,K = \bigoplus \mathcal{F}^m(K)/\mathcal{F}^{m+1}(K) = \bigoplus_{m,j} K^{m,j} \cong K$

  induced $D = d' \Rightarrow$ $\boxed{E_1 = H^*(\mathrm{Gr}\,K, D) = H^*(K, d') = H^*(K, d)}$

- $\underline{d_1}$ Note $\delta : \mathcal{F}^mK/\mathcal{F}^{m+1}K \to \mathcal{F}^{m+1}K/\mathcal{F}^{m+2}K \rightsquigarrow \delta : \mathrm{Gr}\,K \to \mathrm{Gr}\,K$.

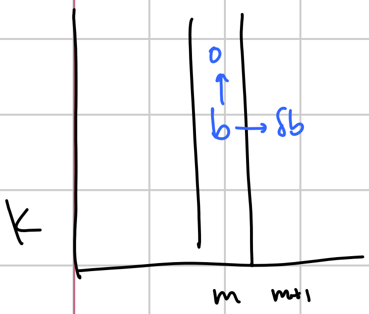
  Since $\delta d = d\delta$, δ descends to $\delta : E_1 \to E_1$.

  $\underline{\text{Claim}}$ $d_1 = \delta$; so $\boxed{E_2 = H^*(H^*(K, d), \delta)}$.

$$\begin{tikzcd} A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd} \qquad \begin{array}{l} A_1 = \bigoplus H(\mathcal{F}^mK) \\ E_1 = \bigoplus H(\mathcal{F}^mK/\mathcal{F}^{m+1}K) \end{array}$$

Write an element of $H^n(\mathcal{F}^mK/\mathcal{F}^{m+1}K, d' = D) \subset E_1$ as $[b]_1$, where $b \in K^{m, n-m}$ with $db = 0$.

(Notation: $b \in K \rightsquigarrow [b]_1 =$ class of b in E_1.)



What's $k_1[b]_1$? $k_1 : H(\mathcal{F}^m/\mathcal{F}^{m+1}) \to H(\mathcal{F}^{m+1})$:

$$\begin{tikzcd}
 & \delta b \arrow[r] & \delta b & & \\
0 \arrow[r] & \mathcal{F}^{m+1}K^{n+1} \arrow[r] & \mathcal{F}^m K^{n+1} \arrow[r] & {} & \\
{} \arrow[rr] & & \mathcal{F}^m K^n \arrow[u, "D"] \arrow[r] & \mathcal{F}^m K^n/\mathcal{F}^{m+1}K^n \arrow[r] & 0 \\
 & & b \arrow[uuu, bend left] \arrow[r] & {[b]} &
\end{tikzcd}$$

So $k_1[b]_1 = [\delta b] \in H^{n+1}(\mathcal{F}^{m+1}K, D) \subset A_1$.

$\nwarrow$ note: $D(\delta b) = d'\delta b = \pm\delta d b = 0$.

$\Rightarrow d_1[b]_1 = j_1 k_1 [b]_1 = [\delta b]_1 \in H^{n+1}(\mathcal{F}^{m+1}K/\mathcal{F}^{m+2}K) \subset E_1$.

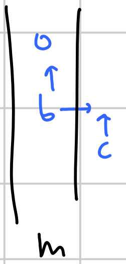
11/11

- $\underline{E_2} = H^*(H^*(K,d),\delta)$.

- $\underline{d_2}$ Represent an element in E_2 as follows:

$b \in K^{m,n-m}$, $db = 0 \rightsquigarrow [b]_1 \in E_1 = H^*(K,d)$

$[\delta b]_1 = 0 \Rightarrow [\delta b] = 0 \in H^*(K,d) \Rightarrow \exists\, c \in K^{m+1,n-m-1}$ with $\delta b + d'c = 0$.



In this case, denote the image of b in E_2 by $[b]_2$.

$$\begin{tikzcd}
A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "j_1"] \\
 & E_1 \arrow[ul, "k_1"] &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
A_2 \arrow[rr, "i_2"] & & A_2 \arrow[dl, "j_2"] \\
 & E_2 = H(E_1, d_1) \arrow[ul, "k_2"] &
\end{tikzcd}$$

recall $j_2(i_1 a) = [j_1 a]$ $a \in A_1$

$k_2[b] = k_1 b$ $b \in E_1$

Now: $[b]_2 \in E_2 \Rightarrow d_2[b]_2 = ?$

$d_2[b]_2 = j_2 k_2 [b]_2 = [j_1 a]_2$ if $a \in A_1$ with $i_1 a = k_1[b]_1$.

$$k_1[b]_1 = \underset{\underset{H(\mathcal{F}^{m+1})}{\uparrow}}{[\delta b]} = i_1(?)$$

$\nwarrow$ need something in $H(\mathcal{F}^{m+2})$

Let's try again to compute $k_1 : H(\mathcal{F}^m/\mathcal{F}^{m+1}) \to H(\mathcal{F}^{m+1})$ for $[b]_1$.

$$\begin{tikzcd}
 & \delta c \arrow[r] & \delta c & & \\
0 \arrow[r] & \mathcal{F}^{m+1} \arrow[r] & \mathcal{F}^m \arrow[r] & {} & \\
{} \arrow[r] & {} & \mathcal{F}^m \arrow[u, "D"] \arrow[r] & \mathcal{F}^m/\mathcal{F}^{m+1} \arrow[r] & 0 \\
 & & b+c \arrow[r] \arrow[uuu, bend left] & {[b]} &
\end{tikzcd}$$

$\Rightarrow k_1[b]_1 = [\delta c] \in H(\mathcal{F}^{m+1}) \subset A_1$

but this is $i_1[\delta c]$ where $[\delta c] \in \underline{H(\mathcal{F}^{m+2})}$.

(note $D(\delta c) = D^2(b+c) = 0$).

So $d_2[b]_2 = [j_1 \underset{\substack{\Uparrow \\ H(\mathcal{F}^{m+2})}}{[\delta c]}]_2 = [\underset{\substack{\Uparrow \\ H(\mathcal{F}^{m+2}/\mathcal{F}^{m+3})}}{\delta c}]_2$.

Note $[\delta c]_2 \in H(H(K,d),\delta)$: $d(\delta c) = \delta d c = \pm\delta^2 b = 0$ and $\delta(\delta c) = 0$.

Conclusion:

$[b]_2 \in E_2 \Rightarrow$
$$\begin{tikzcd}
0 & & \\
b \arrow[u] \arrow[r] & {} & \\
 & c \arrow[u] \arrow[r, dashed] & \delta c
\end{tikzcd}$$
$\to d_2[b]_2 = [\delta c]_2$.

Can check: def of d_2 indep of choice of b, c.

<u>Def</u> $b \in K^{*,*}$ <u>lives to E_r</u> if b descends to a class in E_r:

$[b]_1 \in E_1$, $d_1[b]_1 = 0 \Rightarrow [b]_2 \in E_2$, $d_2[b]_2 = 0 \Rightarrow [b]_3 \in E_3$, $\dots$, $d_{r-1}[b]_{r-1} = 0 \Rightarrow [b]_r \in E_r$.

$$\begin{tikzcd}
0 \\
b \arrow[u]
\end{tikzcd}$$
$\Leftrightarrow$ b lives to E_1, and $d_1[b]_1 = [\delta b]_1$
$$\begin{tikzcd}
0 & \\
b \arrow[u] \arrow[r, dashed, "d_1"] & \delta b
\end{tikzcd}$$

$$\begin{tikzcd}
0 & \\
b \arrow[u] \arrow[r] & {} \\
 & c \arrow[u]
\end{tikzcd}$$
$\Leftrightarrow$ b lives to E_2, and $d_2[b]_2 = [\delta b]_2$
$$\begin{tikzcd}
0 & & \\
b \arrow[u] \arrow[r] & {} & \\
 & c \arrow[u] \arrow[r, dashed, "d_2"] & \delta c
\end{tikzcd}$$

Similarly, b lives to E_3 $\Leftrightarrow$

$$\begin{tikzcd}
0 & & \\
b \arrow[u] \arrow[r] & {} & \\
& c_1 \arrow[u] \arrow[r] & {} \\
& & c_2 \arrow[u]
\end{tikzcd}$$

and $d_3[b]_3 = [\delta c_2]_3$

$$\begin{tikzcd}
0 & & & \\
b \arrow[u] \arrow[r] & {} & & \\
& c_1 \arrow[u] \arrow[r] & {} & \\
& & c_2 \arrow[u] \arrow[r, dashed, "d_3"] & \delta c_2
\end{tikzcd}$$

And in general, b live to $E_r \Leftrightarrow$

$$\begin{tikzcd}
0 & & & & \\
b \arrow[u] \arrow[r] & {} & & & \\
& c_1 \arrow[u] \arrow[r] & {} & & \\
& & c_2 \arrow[u] \arrow[r] & {} & \\
& & & \ddots & \\
& & & c_{r-1} \arrow[u] \arrow[r, dashed, "d_r"] & \delta c_{r-1}
\end{tikzcd}$$

and $d_r[b]_r = [\delta c_{r-1}]_r$.

Give E_r the bigrading inherited from $K^{*,*}$:

$$E_r = \bigoplus_{i,j\geq 0} E_r^{i,j}. \quad (\text{if } b\in K^{i,j} \text{ survive to } E_r \text{ then } [b]_r \in E_r^{i,j}).$$

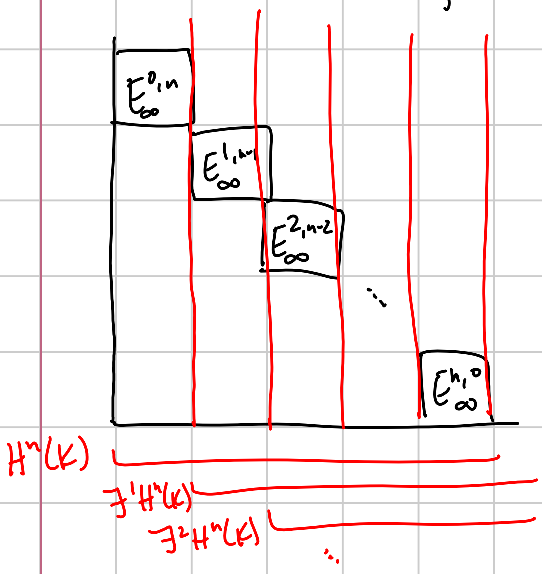
Then note: d_r has bidegree $(r,1-r)$:

$$d_r : E_r^{i,j} \longrightarrow E_r^{i+r,\, j-r+1}.$$

What's E_∞?

$$E_\infty^n = \bigoplus_m \mathcal{F}^m(H^nK)/\mathcal{F}^{m+1}(H^nK) \quad , \quad \mathcal{F}^m(H^nK) = i^m H^n(\mathcal{F}^mK).$$

but also $E_\infty^n = \bigoplus_{i+j=n} E_\infty^{i,j}$.



So $E_\infty^{0,n} \cong H^n(K)/\mathcal{F}^1H^n(K)$

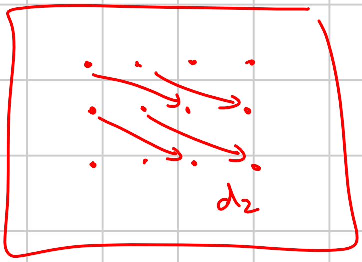
$E_\infty^{1,n-1} \simeq \mathcal{F}^1H^nK/\mathcal{F}^2H^nK$

$\vdots$

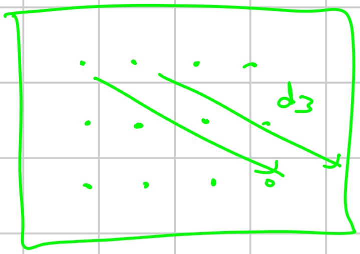
Thm $K^{*,*}$ double complex. There is a spectral sequence $(E_r^{*,*}, d_r)$ with $E_\infty = \mathrm{Gr}\, H(K)$; more specifically,

- d_r has bidegree $(r, 1-r)$
- $E_1^{i,j} = H_d^{i,j}(K)$
- $E_2^{i,j} = H_\delta^{i,j}(H_d(K))$
- $\mathrm{Gr}\, H_D^n(K) \cong \bigoplus_{i+j=n} E_\infty^{i,j}$.

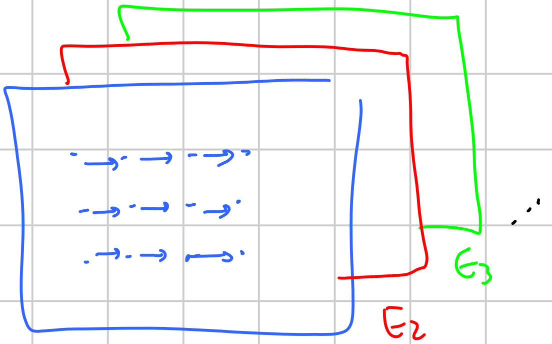
E_1



E_2

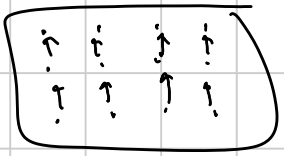


E_3



E_1

“pages” of the spectral sequence (could also throw in



)

E_0

Note Can also do this reflected

So $\exists$ another spectral sequence E_r' with

- d_r has bidegree $(1-r, r)$
- $E_1' = H_\delta(K)$
- $E_2' = H_d H_\delta(K)$
- $E_\infty' = \mathrm{Gr}\, H(K)$ (associated graded for a <u>different</u> filtration on $H(K)$).

Let's apply this to deRham ≃ Čech.

$K^{i,j} = C^i(\mathcal{U}\ \Omega^j)$, $\mathcal{U}$ = good cover.

$$\begin{tikzcd}
\vphantom{X} & \vphantom{X} & \\
C^0(\mathcal{U},\Omega^1) \arrow[r] \arrow[u] & C^1(\mathcal{U},\Omega^1) \arrow[r] \arrow[u] & {} \\
C^0(\mathcal{U},\Omega^0) \arrow[r] \arrow[u] & C^1(\mathcal{U},\Omega^0) \arrow[r] \arrow[u] & {}
\end{tikzcd}
\rightsquigarrow E_1 = H_d K =
\begin{tikzcd}
0 & 0 & \\
0 & 0 & \\
C^0(\mathcal{U},\mathbb{R}) \arrow[r, "d_1=\delta"] & C^1(\mathcal{U},\mathbb{R}) \arrow[r] & {}
\end{tikzcd}$$

$$\rightsquigarrow E_2 = H_\delta H_d K =
\begin{array}{|lll}
0 & 0 & \\
0 & 0 & \\
\check{H}^0(\mathcal{U},\mathbb{R}) & \check{H}^1(\mathcal{U};\mathbb{R}) & \cdots \\
\hline
\end{array}$$

All higher diffs $d_2, d_3, \ldots$ must be 0.

$$\Rightarrow H_D^n(C^*(\mathcal{U},\Omega^*)) \simeq \check{H}^n(\mathcal{U};\underline{\mathbb{R}}).$$

Mayer-Vietoris $\Rightarrow$

$$E_1' = H_\delta(K) =
\begin{tikzcd}
\Omega^2(M) & 0 \\
\Omega^1(M) \arrow[u, "d_1'=d"'] & 0 \\
\Omega^0(M) \arrow[u, "d_1'=d"'] & 0
\end{tikzcd}
\rightsquigarrow E_2' = H_d H_\delta(K) =
\begin{array}{|ll}
\vdots & \\
H^2_{DR}(M) & 0 \\
H^1_{DR}(M) & 0 \\
H^0_{DR}(M) & 0 \\
\hline
\end{array}$$

$$\Rightarrow \quad H_D^n(C^*(\mathcal{U},\Omega^*)) \simeq H^n_{DR}(M)$$

$$\Rightarrow \quad H^*_{DR}(M) \simeq \check{H}^*(\mathcal{U};\underline{\mathbb{R}}).$$

Leray-Serre Spectral Sequence

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$$ fiber bundle, $\mathcal{U}$ = good cover of B

$\pi^{-1}(\mathcal{U})$ = cover of E.

Consider the double complex

$$K^{i,j} = C^i(\pi^{-1}\mathcal{U}, \Omega^j) = \prod_{\alpha_0<\cdots<\alpha_i} \Omega^j(\pi^{-1}(U_{\alpha_0\cdots\alpha_i})).$$

MV sequence is exact ∀ open covers ⇒

$$0 \to \Omega^j(E) \to C^0(\pi^{-1}(\mathcal{U}), \Omega^j) \to C^1(\pi^{-1}(\mathcal{U}), \Omega^j) \to \cdots$$ is exact.

$$\Rightarrow \quad E_1 = \begin{array}{|l} \Omega^2(E) \\ \Omega^1(E) \\ \Omega^0(E) \\ \hline \end{array} \Rightarrow E_2 = \begin{array}{|l} H^2(E) \\ H^1(E) \\ H^0(E) \\ \hline \end{array} \Rightarrow H^*(K, D) \cong H^*_{DR}(E).$$

Other spectral sequence? Note $\pi^{-1}(\overset{\text{if nonempty}}{U_{\alpha_0\cdots\alpha_i}}) \cong \mathbb{R}^n \times F \Rightarrow$

i^{th} column is

$$\begin{array}{ccc} \vdots & & \\ \Omega^2(\pi^{-1}U_{\alpha_0\cdots\alpha_i}) & & H^2(\quad) \\ \Omega^1(\pi^{-1}U_{\alpha_0\cdots\alpha_i}) & \xrightarrow{H_*} & H^1(\quad) \\ \Omega^0(\pi^{-1}U_{\alpha_0\cdots\alpha_i}) & & H^0(\pi^{-1}U_{\alpha_0\cdots\alpha_i}) \end{array}$$

so $E_1^{i,j} = C^i(\mathcal{U}, \mathcal{H}^j)$ where $\mathcal{H}^*(U) = H^*(\pi^{-1}(U))$.

$\Rightarrow E_2^{i,j} = H_\delta^{i,j}(C^*(\mathcal{U}, \mathcal{H}^*)) = \check{H}^i(\mathcal{U}, \mathcal{H}^j)$.

LERAY-SERRE S.S. →

Thm (Leray) $\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd}$, $\mathcal{U}$ = good cover of B. Then $\exists$ spectral sequence converging to $H^*_{DR}(E)$ with E_2 page $E_2^{i,j} = \check{H}^*(\mathcal{U}, \mathcal{H}^j)$.

If B is simply connected, then

$$E_2^{i,j} \cong H^i_{DR}(B) \otimes H^j_{DR}(F).$$

Pf of last stmt: Write $H^j_{DR}(F) = \mathbb{R}^{b_j(F)}$; then $\mathcal{H}^j \cong \underline{\mathbb{R}^{b_j(F)}}$ constant presheaf, since locally const ⇒ const.

$$\Rightarrow E_2^{i,j} = \check{H}^i(\mathcal{U}, \underline{\mathbb{R}^{b_j(F)}}) \cong \bigoplus_{b_j(F) \text{ copies}} \check{H}^i(\mathcal{U}, \underline{\mathbb{R}})$$

$$\cong \bigoplus_{b_j(F)} H^i_{DR}(B)$$

$$\cong H^i_{DR}(B) \otimes H^j_{DR}(F). \qquad \square$$

Note Holds more generally for Serre fibrations (fibers are only htpy equiv; satisfies htpy lifting property) – useful for calculating higher htpy groups.

Can use Leray-Serre to calculate $H^*(E)$ given $H^*(B)$, $H^*(F)$. But can also switch things around in some cases!

Ex $\mathbb{C}P^2$. Fiber bundle

$$\begin{tikzcd} S^1 \arrow[r] & S^5 \arrow[d] \\ & \mathbb{C}P^2 \end{tikzcd}$$

Note $\mathbb{C}P^2 = (0\text{-cell}) \cup (2\text{-cell}) \cup (4\text{-cell})$ is simply connected.

$\to \exists$ spectral sequence converging to $H^*(S^{2n+1})$

$E_2^{i,j} = H^i(\mathbb{C}P^2) \otimes H^j(S^1)$. Suppose we don't know $H^*(\mathbb{C}P^2)$.

E_2:

$$\begin{tikzcd} \mathbb{R} \arrow[rrd] & ?_1 \arrow[rrd] & ?_2 \arrow[rrd, "d_2"] & ?_3 & ?_4 \\ \mathbb{R} & ?_1 & ?_2 & ?_3 & ?_4 \end{tikzcd}$$

$d_k = 0$ for $k \geq 3 \Rightarrow E_3 = E_\infty$

$$H^*(S^5) = \begin{Bmatrix} \mathbb{R} & * = 0, 5 \\ 0 & \text{otherwise} \end{Bmatrix} = E_3^* \Rightarrow ?_1 = 0,\ ?_4 = \mathbb{R},\ ?_3 = 0$$

$$\begin{tikzcd} \mathbb{R} \arrow[rrd] & 0 & ?_2 \arrow[rrd] & 0 & \mathbb{R} \\ \mathbb{R} & 0 & ?_2 & 0 & \mathbb{R} \end{tikzcd}$$

$\Rightarrow ?_2 = \mathbb{R}$

$$\therefore H^*(\mathbb{C}P^2) = \begin{cases} \mathbb{R}, & * = 0, 2, 4 \\ 0, & \text{otherwise.} \end{cases}$$

Note: works more generally to compute $H^*(\mathbb{C}P^n)$.

---

recall

Leray-Hirsch Thm $F \to E$, $\downarrow$ B, $\mathcal{U}$ = good cover, rk $H^*(F) = r$.

Suppose $\exists$ $e_1, \ldots, e_r \in \Omega^*(E)$ global forms representing cohom classes s.t. restricted to $E_x \cong F$, these generate $H^*(F)$ $\forall x \in B$. Then

$$H^*(E) \cong H^*(B) \otimes H^*(F).$$

(in particular, Künneth Thm).

Pf Spectral sequence for $\underbrace{C^i(\pi^{-1}(\mathcal{U}),\Omega^j)}_{\text{double complex}}$ with $E_2^{i,j} = \check{H}^i(\mathcal{U},\mathcal{H}^j)$

and $E_\infty \cong H^*(E)$. Note

$\mathcal{H}^*(U) = \underbrace{H^*(\pi^{-1}(U))}_{\text{basis } [e_1|_{\pi^{-1}(U)}],\dots,[e_r|_{\pi^{-1}(U)}]} \cong H^*(F) \cong \mathbb{R}^r$ and if $V\subset U$,

$$\begin{tikzcd} \mathcal{H}^*(U) \arrow[rr, "r"] \arrow[dr, "\cong"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathcal{H}^*(V) \arrow[dl, "\cong"] \\ & \mathbb{R}^r & \end{tikzcd}$$

So $\mathcal{H}^*$ is constant and

$$E_2^{i,j} \cong \check{H}^i(\mathcal{U},\mathcal{H}^j) \cong \check{H}^i(\mathcal{U},\mathbb{R})\otimes H^j(F) \cong H^i(B)\otimes H^j(F).$$

Claim: $d_2=d_3=\cdots=0$ so $E_2=E_\infty$.

Pf: An elt in $E_2^{i,j}$ comes from $C^i(\pi^{-1}(\mathcal{U}),\Omega^j) = \prod \Omega^j\pi^{-1}(U_{\alpha_0\cdots\alpha_i})$.

$\overset{\shortparallel}{\check{H}^i(\mathcal{U},\mathbb{R})\otimes H^j(F)}$

Say $[\omega]\otimes[\tau] \in \check{H}^i(\mathcal{U},\mathbb{R})\otimes H^j(F)$, $\omega\in C^i(\mathcal{U},\mathbb{R})$ with $\delta\omega=0$, $d\tau=0$.

Where does this sit in the double complex? It's represented by

$\omega\otimes\tilde{\tau}$, $\tilde{\tau}\in\Omega^j E$ linear comb of $e_1,\dots,e_r$ corr. to $[\tau]\in H^j(F)$.

i.e. $(\omega\otimes\tilde{\tau})_{\alpha_0\cdots\alpha_i} = \omega_{\alpha_0\cdots\alpha_i}\tilde{\tau}|_{\pi^{-1}(U_{\alpha_0\cdots\alpha_i})}$.

So $d(\omega\otimes\tilde{\tau})=0$, $\delta(\omega\otimes\tilde{\tau})=0$.

$$\begin{tikzcd} 0 & \\ \omega\otimes\tilde{\tau} \arrow[u, "d"] \arrow[r, "\delta"] & 0 \end{tikzcd}$$

$\Rightarrow d_2(\omega\otimes\tilde{\tau})=d_3(\omega\otimes\tilde{\tau})=\cdots=0.$ □

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Product Structure and Spectral Sequences

Recall $C^*(\mathcal{U},\Omega^*)$ has a product structure: $\omega\in C^{k_1}(\mathcal{U},\Omega^{\ell_1})$, $\eta\in C^{k_2}(\mathcal{U},\Omega^{\ell_2})$

$\rightsquigarrow (\omega\cup\eta)\in C^{k_1+k_2}(\mathcal{U},\Omega^{\ell_1+\ell_2})$

$$(\omega\cup\eta)_{\alpha_0\cdots\alpha_{k_1+k_2}} = (-1)^{\ell_1 k_2}\,\omega_{\alpha_0\cdots\alpha_{k_1}}\wedge\eta_{\alpha_{k_1}\cdots\alpha_{k_1+k_2}}.$$

In general, we say $(K^{*,*}, \delta, d)$ has a <u>product structure</u> if $\exists$ map $\cup : K^{k_1,l_1} \otimes K^{k_2,l_2} \to K^{k_1+k_2, l_1+l_2}$ s.t.

$$D(\omega \cup \eta) = (D\omega) \cup \eta + (-1)^{|\omega|} \omega \cup (D\eta)$$

$|\omega| = k_1 + l_1$

<u>D is an antiderivation wrt $\cup$</u>

δ ---

d' --

(equiv: $\delta(\omega \cup \eta) = (\delta\omega) \cup \eta + (-1)^{|\omega|} \omega \cup (\delta\eta)$

$d'(\omega \cup \eta) = (d'\omega) \cup \eta + (-1)^{|\omega|} \omega \cup (d'\eta)$)

In this case, $\cup$ descends to $E_1 = H_d K$ and $d_1 = \delta$ is an antiderivation wrt $\cup$

$\to$ $\cup$ descends to $E_2 = H_\delta K$ and d_2 is an antiderivation wrt $\cup$

$\to$ $\cup$ --- E_3 -- d_3 -- - —

$\to$ etc.

<u>Thm</u> $(K^{*,*}, \delta, d)$ double cx with product str. Then (E_r, d_r) inherits a product str wrt which d_r is an antiderivation.

<u>CAUTION</u>: $E_\infty = \mathrm{Gr}\, H_D(K)$ has a product structure, but in general it's different from the product structure on $H_D(K)$.

In Leray-Serre s.s., the product structure on E_2 is the usual product str on $H^*(B) \otimes H^*(F)$:

$$(a \otimes b) \cup (c \otimes d) = (-1)^{|b||c|} (a \cup c) \otimes (b \cup d).$$ (exercise)

We can actually use this to compute the ring structure on $H^*(\mathbb{CP}^n)$! (without invoking Poincaré duality)

<u>Ex</u> $H^*(\mathbb{CP}^2)$.

$$\begin{tikzcd} S^1 \arrow[r] & S^5 \arrow[d] \\ & \mathbb{CP}^2 \end{tikzcd}$$

$$\begin{tikzcd} & & a\cup x \arrow[d, dash, dashed] & & \\ a\in \mathbb{R} \arrow[drr, "d_2"] & 0 & \mathbb{R} \arrow[drr, "d_2"] & 0 & \mathbb{R} \\ \mathbb{R} & 0 & \mathbb{R} \arrow[d, dash, dashed] & 0 & \mathbb{R} \\ & & x = d_2 a & & \end{tikzcd}$$

$E_2 = H^*(\mathbb{C}P^2) \otimes H^*(S^1)$

Say a generates $H^1(S^1)$

$\Rightarrow d_2 a =: x$ generates $H^2(\mathbb{C}P^2)$

$\Rightarrow a \cup x$ generates $H^1(S^1) \otimes H^2(\mathbb{C}P^2) = E_2^{2,1}$

$\Rightarrow d_2(a \cup x) = (d_2 a) \cup x = x \cup x \neq 0$

$\therefore x^2 \neq 0$ in $H^*(\mathbb{C}P^2) \rightarrow H^*(\mathbb{C}P^2) \cong \mathbb{R}[x]/(x^3)$.

Leray-Serre with integer coefficients

Instead of using the presheaf Ω^* over M, use the presheaf S^* of singular cochains:

$$S^j(U) = \{\text{singular cochains on } U\} = \{\varphi : \underset{\substack{\uparrow \\ \text{singular } j\text{-simplex}}}{\sigma} \to \mathbb{Z}\} = C^j_{\text{sing}}(U, \mathbb{Z})$$

$\mathcal{U}$ = open cover $\rightsquigarrow$ $K^{i,j} = C^i(\mathcal{U}, S^j)$

$$\begin{tikzcd} C^0(\mathcal{U}, S^2) & C^1(\mathcal{U}, S^2) & \\ C^0(\mathcal{U}, S^1) \arrow[r] \arrow[u] & C^1(\mathcal{U}, S^1) \arrow[r] \arrow[u] & {} \\ C^0(\mathcal{U}, S^0) \arrow[r, "\delta"] \arrow[u, "\delta"] & C^1(\mathcal{U}, S^0) \arrow[r] \arrow[u] & {} \end{tikzcd}$$

Generalized MV:

$$0 \to \underset{\substack{\uparrow \\ \text{singular cochains from simplices in } \mathcal{U} \text{ to } \mathbb{Z}}}{S^j_{\mathcal{U}}(M, \mathbb{Z})} \to C^0(\mathcal{U}, S^j) \to C^1(\mathcal{U}, S^j) \to \cdots \text{ is exact}$$

$$E_1 = H_\delta K = \left| \begin{array}{l} S^2_{\mathcal{U}}(M,\mathbb{Z}) \\ S^1_{\mathcal{U}}(M,\mathbb{Z}) \\ S^0_{\mathcal{U}}(M,\mathbb{Z}) \\ \hline \end{array} \right. \rightarrow E_2 = \left| \begin{array}{l} H^2(M,\mathbb{Z}) \\ H^1(M,\mathbb{Z}) \\ H^0(M,\mathbb{Z}) \\ \hline \end{array} \right.$$

(actually $H^*_{\mathcal{U}}(M,\mathbb{Z})$ but this is the same thing)

If $\mathcal{U}$ = good cover then $E_1 = H_{\delta_1}(K) = \begin{array}{|lll} & & \\ \check{C}^0(\mathcal{U},\mathbb{Z}) & \check{C}^1(\mathcal{U},\mathbb{Z}) & \cdots \\ \hline \end{array} \Rightarrow E_2 = \begin{array}{|lll} & & \\ \check{H}^0(\mathcal{U},\mathbb{Z}) & \check{H}^1(\mathcal{U},\mathbb{Z}) & \cdots \\ \hline \end{array}$

Good covers are cofinal

$\Rightarrow \quad H^*_{sing}(M,\mathbb{Z}) \cong \check{H}^*(M,\mathbb{Z})$.

Same argument as Leray-Serre but with S^* instead of Ω^*:

<u>Thm</u> (Leray) $\begin{array}{c} F \to E \\ \downarrow \\ B \end{array}$ fiber bundle. Assume B simply connected, $\mathcal{U}$ = good cover.

Then $\exists$ spectral sequence (E_r, d_r) converging to $\operatorname{Gr} H^*_{sing}(E,\mathbb{Z})$ with E_2 term

$$E_2^{i,j} = H^i(B, H^j(F,\mathbb{Z}));$$

if $H^*(F,\mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module, then

$$E_2^{i,j} = H^i(B,\mathbb{Z}) \otimes H^j(F,\mathbb{Z}).$$

Gysin Sequence

Suppose we have a <u>sphere bundle</u>: $\begin{array}{c} S^k \to E \\ \downarrow \\ B \end{array}$ for some k.

Also suppose $\mathcal{H}^*$ is <u>constant</u> – happens if $\pi_1(B) = 1$, or if the sphere bundle is <u>oriented</u>: all transition fns $\in \operatorname{Diff}(S^k)$ are orient-preserving.

Then

$$E_2 = \begin{tikzcd} k & H^0(B) \arrow[ddrr] & H^1(B) & H^2(B) & \cdots \\ & & & & \\ 0 & H^0(B) & H^1(B) & H^2(B) & \cdots \end{tikzcd}$$

d_{k+1} = only possible nonzero differential

$$\begin{tikzcd} E_2^{n-k,k} = H^{n-k}(B) \arrow[dr, "d_{k+1}"] & \\ & E_2^{n+1,0} = H^{n+1}(B) \end{tikzcd}$$

$\Rightarrow$ exact seq.

$$\boxed{0 \to E_\infty^{n-k,k} \to H^{n-k}(B) \xrightarrow{d_{k+1}} H^{n+1}(B) \to E_\infty^{n+1,0} \to 0}$$

Now $H^n(E) = E_\infty^n = E_\infty^{n-k,k} \oplus E_\infty^{n,0}$ so we can sum these exact sequences to get the Gysin sequence

$$\boxed{\cdots \to H^n(E) \to H^{n-k}(B) \to H^{n+1}(B) \to H^{n+1}(E) \to \cdots}$$

The interesting thing: we know what these maps are.

π_* = integration along the fiber

$\wedge e$, $e \in H^{k+1}(M)$ Euler class of E

π^*, $\pi : E \to B$

Ex $k=1$, $E = S^5$, $B = \mathbb{C}P^2$. 0

$$\begin{tikzcd} 0 \arrow[r] & \overset{\mathbb{R}}{\boxed{H^5(E)}} \arrow[r] & H^4(B) \arrow[dll] \\ H^4(B) \arrow[r] & \cancel{H^4(E)} \arrow[r] & H^3(B) \arrow[dll] \\ H^3(B) \arrow[r] & \cancel{H^3(E)} \arrow[r] & H^2(B) \arrow[dll] \\ H^2(B) \arrow[r] & \cancel{H^2(E)} \arrow[r] & H^1(B) \arrow[dll] \\ H^1(B) \arrow[r] & \cancel{H^1(E)} \arrow[r] & H^0(B) \arrow[dll] \\ H^0(B) & \underset{\mathbb{R}}{\boxed{H^0(E)}} & 0 \end{tikzcd}$$

This recovers $H^*(\mathbb{C}P^2)$.